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Composing cardinal direction relations ☆

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Abstract

We study the recent proposal of Goyal and Egenhofer who presented a model for qualitative spatial reasoning about cardinal directions. Our approach is formal and complements the presentation of Goyal and Egenhofer. We focus our efforts on the composition operator for two cardinal direction relations. We consider two interpretations of the composition operator: consistency-based and existential composition. We point out that the only published method to compute the consistency-based composition does not always work correctly. Then, we consider progressively more expressive classes of cardinal direction relations and give consistency-based composition algorithms for these classes. Our theoretical framework allows us to prove formally that our algorithms are correct. When we consider existential composition, we demonstrate that the binary relation resulting from the composition of two cardinal direction relations cannot be expressed using the relations defined by Goyal and Egenhofer. Finally, we discuss some extensions to the basic model and consider the composition problem for these extensions.

Keywords: Cardinal direction relations; Spatial constraints; Consistency-based composition; Existential composition; Qualitative spatial reasoning; Composition table

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1. Introduction

Qualitative spatial reasoning has received a lot of attention in the areas of Geographic Information Systems [13,15], Artificial Intelligence [6,14,36,38], Databases [34] and Multimedia [41]. Several kinds of useful spatial relations have been studied so far, e.g., topological relations [4,6,13,36,38], cardinal direction relations [19,25,29] and qualitative distance relations [15,44].

In this paper, we concentrate on *cardinal direction relations* [19,25,32,39] which are used to describe how regions of space are placed relative to one another (e.g., region a is *north of* region b). Our starting point is the recent model of Goyal and Egenhofer [18,19]. This model is currently one of the most expressive models for qualitative reasoning with cardinal directions. It works with extended regions and has potential in Multimedia and Geographic Information Systems applications.

In our work we focus on the problem of *composing* cardinal direction relations. The composition operation for various kinds of spatial relations has received a lot of attention in the literature [13,15,17,21,32,38]. Research has mainly concentrated on two definitions of composition, namely *existential* and *consistency-based* composition [5,26]. Existential composition is the standard definition of composition from set theory [5,26,32, 38]. Consistency-based composition is a weaker interpretation useful in several domains [11,19,35]. Both definitions, are typically used as mechanisms for inferring new spatial relations from existing ones. This is very helpful in the following situations:

(1) Detecting possible inconsistencies in a given set of spatial relations [25,38].
(2) Preprocessing spatial queries so that inconsistent queries are detected or the search space is pruned [31].

Therefore, it is very important to study the composition operation and define appropriate composition tables for interesting models of spatial relations such as the model of Goyal and Egenhofer [18,19].

The technical contributions of this paper can be summarized as follows:

- We give formal definitions for the cardinal direction relations that can be expressed in the model of Goyal and Egenhofer [19].
- We use our formal framework to study the composition operation for cardinal direction relations in the model of [19]. Initially, we consider consistency-based composition. Goyal and Egenhofer first studied this operation in [19] but their method does not always work correctly.
- The previous observation leaves us with the task of finding a correct method for computing the consistency-based composition. To do this, we consider progressively more expressive classes of cardinal direction relations and give consistency-based composition algorithms for these classes. Our theoretical framework allows us to *prove formally* that our algorithms are correct.
- Then, we consider the existential definition of composition and we show something very interesting: the binary relation resulting from the existential composition of some cardinal direction relations *cannot even be expressed* using the relations defined in

[19] unless the language is augmented by extra predicates (Section 6). This non-expressibility result does not allow us to use existential composition to infer new information. The above should be contrasted to the consistency-based definition which is expressible for any pair of cardinal direction relations and thus can be used as a constraint propagation mechanism.
- Finally, we discuss some extensions to the basic model and consider the composition problem for these extensions.

In a recent paper [40] we have studied the problem of checking the consistency of a given set of constraints in the model of [18,19] and its variations.

The rest of the paper is organized as follows. In Section 2, we survey related work. Section 3 presents the formal model. In Sections 4 and 5, we consider classes of cardinal direction relations and give consistency-based composition algorithms for these classes. Section 6 shows that the result of existential composition of cardinal direction relations cannot be expressed using the relations defined in [19]. In Section 7, we summarize our results. Section 8 presents some extensions to the basic model. Our conclusions are presented in Section 9.

2. Related work

Qualitative spatial reasoning forms an important part of the commonsense reasoning required for building successful intelligent systems [8]. Most researchers in qualitative spatial reasoning have dealt with three main classes of spatial information: topological, directional and distance. *Topological* relations describe how the boundaries, the interiors and the exteriors of two regions relate [4,6,13,36–38]. For instance, if a and b are regions then *a includes b* and *a externally connects with b* are topological constraints. *Directional* (or *orientation*) relations describe where regions are placed relative to one another. For instance, *a north b* and *a southeast b* are directional constraints [1,14,17,19,25,32,39]. Finally, *distance* relations describe the relative distance of two regions. For instance, *a is far from b* and *a is close to b* are distance constraints [15,44].

Several definitions of the composition operator had been studied for all the above classes of spatial relations. Particularly, the consistency-based and existential definition of the composition operator have attracted the interest of researchers [5,10,11,19,21,26,28,32,35,38]. Typically, the above definitions are used as mechanisms for inferring new relations from existing ones. Such inference mechanisms are very important as they are in the heart of any system that retrieves collections of objects similarly related to each other using spatial relations [31]. Composition is also an essential part of Relation Algebras [42,43]. Specifically, for qualitative spatial reasoning, Relation Algebras have been studied in a great extent [9,10,28]. Moreover, composition is used to identify classes of relations that have a tractable consistency problem [21,30,38].

Several models capturing cardinal directions have been proposed in the literature. Most models approximate a region by a point or its minimum bounding box. Point-based models [16,17,25] use a representative point (most commonly the centroid) to approximate regions. Given two regions a and b, the point-based models divide the space around the

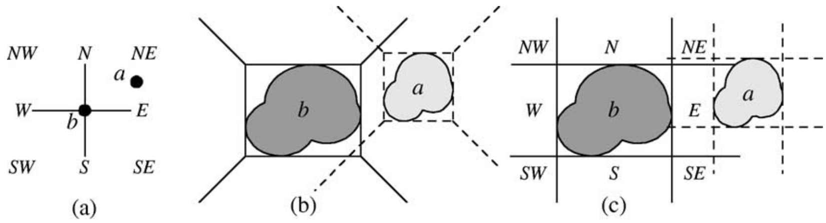

Fig. 1. Directional models.

representative point region b into a number of mutually exclusive areas. The area where the representative point of region a lies characterizes its directional relation with respect to region b. For example, in Fig. 1(a) we have four mutually exclusive areas of dimension 2, four semi-axes and a point. For the points in this figure the constraint *a northeast b* holds.

Papadias [32] represents each region by two points in $\mathbb{R}^2$ which correspond to the lower-left and the upper-right corner of the region's minimum bounding box. Directional and topological relations are defined on minimum bounding boxes. Such relations are described by applying Allen's [2] interval relations along the projections of regions to the x- and y-axes.

Abdelmoty [1,3] extends the 4-intersection formalism [12] for topological relations to represent directions. The space external to the region is divided into four semi-infinite areas by lines starting from the corners of the region's minimum bounding box (Fig. 1(b)). The directional relation between two regions is defined using the intersections of the components of these areas.

More interestingly, to characterize the cardinal direction relation of a region a with respect to a region b, [18,19] partition the space around region b into nine areas (Fig. 1(c)) and record the partitions where region a falls. This gives a cardinal direction constraint between two regions. At a finer level of granularity the model of [18,19] also offers the option to record how much of region a falls into each partition of region b. For example region *a is* 90% *east and* 10% *northeast of region b* (Fig. 1(c)). This model will be presented in Section 3 because it is the main subject of this paper.

Isli and Cohn [22] define relation algebras for cyclic ordering of 2-dimensional orientations. Distances and orientation are combined in a qualitative reasoning framework in [44]. A different approach, coming from the pictorial database area, is presented in [41]. The authors consider topological and directional spatial relations and present a semantics and a corresponding set of inference rules. The set of inference rules is sound and complete with respect to the semantics when we are in $\mathbb{R}^3$ but it is incomplete for $\mathbb{R}^2$.

3. A formal model for cardinal direction information

We consider the Euclidean space $\mathbb{R}^2$. *Regions* are defined as non-empty and bounded sets of points in $\mathbb{R}^2$. Let a be a region. The *greatest lower bound* or the *infimum* [27] of the *projection* of region a on the x-axis (respectively y-axis) is denoted by $inf_x(a)$ (respectively $inf_y(a)$). The *least upper bound* or the *supremum* of the *projection* of region

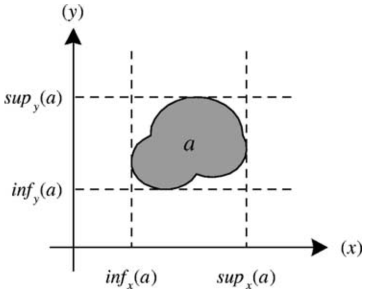

Fig. 2. A region and its bounding box.

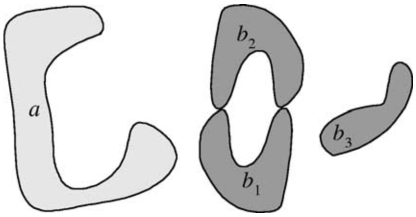

Fig. 3. Regions.

a on the x-axis (respectively y-axis) is denoted by $sup_x(a)$ (respectively $sup_y(a)$). We will often refer to *sup* and *inf* as *endpoints*.

The *minimum bounding box* of a region a, denoted by $mbb(a)$, is the box formed by the straight lines $x = inf_x(a)$, $x = sup_x(a)$, $y = inf_y(a)$ and $y = sup_y(a)$ (see Fig. 2). Obviously, the projections on the x-axis (respectively y-axis) of a region and its minimum bounding box have the same endpoints.

We will consider throughout the paper the following types of regions:

- Regions that are homeomorphic to the *closed unit disk* ($\{(x, y): x^2 + y^2 \leqslant 1\}$). The set of these regions will be denoted by *REG*. Regions in *REG* are *closed, connected* and have *connected boundaries* (for definitions see [7,27]). Notice that the results of Sections 4, 5 and 6 are not affected if we consider regions that are homeomorphic to the *open unit disk* (as in [33]).
- Regions that are formed by finite unions of regions in *REG*. The set of these regions will be denoted by REG^*. Notice that regions in REG^* can be *disconnected* and can have *holes*.

In Fig. 3, regions a, b_1, b_2 and b_3 are in *REG* (also in REG^*) and region $b = b_1 \cup b_2 \cup b_3$ is in REG^*.

Regions in *REG* have been previously studied in [19,33]. They can be used to model areas in various interesting applications, e.g., land parcels in Geographic Information Systems [13,15]. In the rest of this section, we will formally define cardinal direction relations for regions in *REG*. To this end, we will divide a region using lines parallel to the axes (see also Fig. 4). Such divisions generally result in disconnected regions in REG^*

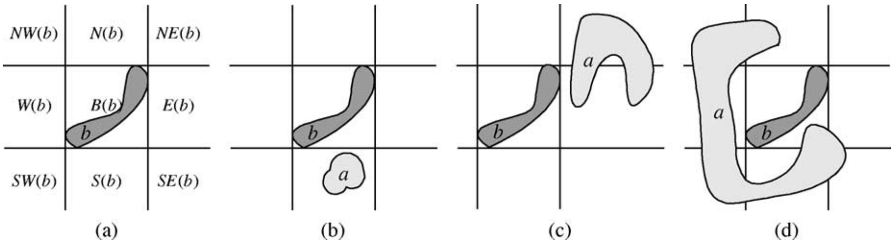

Fig. 4. Reference tiles and relations.

(Fig. 6). Therefore, the definitions of cardinal directions relations for regions in *REG* will also involve regions in REG^* (see Section 3.1 for more details).

Let us now consider two arbitrary regions a and b in *REG*. Let region a be related to region b through a cardinal direction relation (e.g., a is north of b). Region b will be called the *reference* region (i.e., the region *to* which the relation is described) while region a will be called the *primary* region (i.e., the region *from* which the relation is described).[1] The axes forming the minimum bounding box of the reference region b divide the space into 9 areas which we call *tiles* (Fig. 4(a)). The peripheral tiles correspond to the eight cardinal direction relations *south*, *southwest*, *west*, *northwest*, *north*, *northeast*, *east* and *southeast*. These tiles will be denoted by $S(b)$, $SW(b)$, $W(b)$, $NW(b)$, $N(b)$, $NE(b)$, $E(b)$ and $SE(b)$ respectively. The central area corresponds to the region's minimum bounding box and is denoted by $B(b)$. By definition each one of these tiles includes the parts of the axes forming it. Notice that tiles $S(b)$, $SW(b)$, $W(b)$, $NW(b)$, $N(b)$, $NE(b)$, $E(b)$ and $SE(b)$ are unbounded, tile $B(b)$ is bounded and the union of all 9 tiles is $\mathbb{R}^2$.

If a primary region a is included (in the set-theoretic sense) in tile $S(b)$ of some reference region b (Fig. 4(b)) then we say that *a is south of b* and we write $a\ S\ b$. Similarly, we can define *southwest* (SW), *west* (W), *northwest* (NW), *north* (N), *northeast* (NE), *east* (E), *southeast* (SE) and *bounding box* (B) relations.

If a primary region a lies partly in the area $NE(b)$ and partly in the area $E(b)$ of some reference region b (Fig. 4(c)) then we say that *a is partly northeast and partly east of b* and we write $a\ NE{:}E\ b$.

In this paper we would first like to work only with constraints $a\ R\ b$ where a and b are connected regions in *REG*. Thus, we will not deal with cases such us the one in Fig. 5 and we will not have corresponding constraints such us *a is partly north and partly south of b*. Later on, in Section 8, we will also handle disconnected regions.

Let us now capture formally what we have described informally above and define the concept of basic cardinal direction relations.

Definition 1. A *basic cardinal direction relation* is an expression $R_1{:}\cdots{:}R_k$ where

(i) $1 \leqslant k \leqslant 9$,

[1] The terms primary/reference are widely used in relevant literature [13,32,34]. Other researchers might prefer *destination* or *target* instead of primary and *origin* or *base* instead of reference.

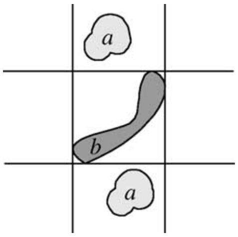

Fig. 5. Disconnected regions.

(ii) $R_1, \ldots, R_k \in \{B, S, SW, W, NW, N, NE, E, SE\}$,
(iii) $R_i \neq R_j$ for every i, j such that $1 \leqslant i, j \leqslant k$ and $i \neq j$, and
(iv) there exist regions $a_1, \ldots, a_k \in REG$ such that $a_1 \in R_1(b), \ldots, a_k \in R_k(b)$ and $a_1 \cup \cdots \cup a_k \in REG$ for any reference region $b \in REG$.

A basic cardinal direction relation $R_1{:}\cdots{:}R_k$ is called *single-tile* if $k = 1$; otherwise it is called *multi-tile*.

Notice that condition (iv) of the above definition is very crucial since it guarantees that a cardinal direction relation is realizable between connected regions. Consider the following example.

Example 1. The following are basic cardinal direction relations:

$$S, \quad NE{:}E \quad \text{and} \quad B{:}S{:}SW{:}W{:}NW{:}N{:}E{:}SE.$$

The first relation is single-tile while the others are multi-tile. Regions involved in these relations are shown in Figs. 4(b), 4(c) and 4(d) respectively.

On the other hand, expression $S{:}N$ is not a basic cardinal direction relation since condition (iv) of Definition 1 is violated. More specifically, for any reference region b we cannot find regions $a_1, a_2 \in REG$ such that $a_1 \in N(b)$, $a_2 \in S(b)$ and $a_1 \cup a_2 \in REG$.

In order to avoid confusion, we will write the single-tile elements of a cardinal direction relation according to the following order: B, S, SW, W, NW, N, NE, E and SE. Thus, we always write $B{:}S{:}W$ instead of $W{:}B{:}S$ or $S{:}B{:}W$. The readers should also be aware that for a basic relation such as $B{:}S{:}W$ we will often refer to B, S and W as its *tiles*.

3.1. Defining basic cardinal direction relations formally

Now we can formally define the single-tile cardinal direction relations B, S, SW, W, NW, N, NE, E and SE of the model as follows:

$$a\ B\ b \quad \text{iff} \quad inf_x(b) \leqslant inf_x(a),\ sup_x(a) \leqslant sup_x(b),\ inf_y(b) \leqslant inf_y(a) \text{ and } sup_y(a) \leqslant sup_y(b).$$

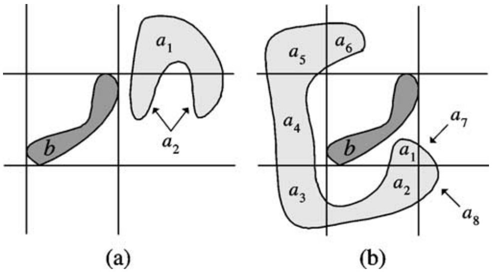

Fig. 6. Relations and component variables.

$a\ S\ b$	iff	$sup_y(a) \leqslant inf_y(b),\ inf_x(b) \leqslant inf_x(a)$ and $sup_x(a) \leqslant sup_x(b)$.
$a\ SW\ b$	iff	$sup_x(a) \leqslant inf_x(b)$ and $sup_y(a) \leqslant inf_y(b)$.
$a\ W\ b$	iff	$sup_x(a) \leqslant inf_x(b),\ inf_y(b) \leqslant inf_y(a)$ and $sup_y(a) \leqslant sup_y(b)$.
$a\ NW\ b$	iff	$sup_x(a) \leqslant inf_x(b)$ and $sup_y(b) \leqslant inf_y(a)$.
$a\ N\ b$	iff	$sup_y(b) \leqslant inf_y(a),\ inf_x(b) \leqslant inf_x(a)$ and $sup_x(a) \leqslant sup_x(b)$.
$a\ NE\ b$	iff	$sup_x(b) \leqslant inf_x(a)$ and $sup_y(b) \leqslant inf_y(a)$.
$a\ E\ b$	iff	$sup_x(b) \leqslant inf_x(a),\ inf_y(b) \leqslant inf_y(a)$ and $sup_y(a) \leqslant sup_y(b)$.
$a\ SE\ b$	iff	$sup_x(b) \leqslant inf_x(a)$ and $sup_y(a) \leqslant inf_y(b)$.

Using the above single-tile relations we can define all multi-tile ones. For instance relation $NE{:}E$ (Fig. 6(a)) and relation $B{:}S{:}SW{:}W{:}NW{:}N{:}E{:}SE$ (Fig. 6(b)) are defined as follows:

$a\ NE{:}E\ b$ iff there exist regions a_1 and a_2 in REG^* such that $a = a_1 \cup a_2$, $a_1\ NE\ b$ and $a_2\ E\ b$.

$a\ B{:}S{:}SW{:}W{:}NW{:}N{:}SE{:}E\ b$ iff there exist regions $a_1, \dots, a_8$ in REG^* such that $a = a_1 \cup a_2 \cup a_3 \cup a_4 \cup a_5 \cup a_6 \cup a_7 \cup a_8,\ a_1\ B\ b,\ a_2\ S\ b,\ a_3\ SW\ b, a_4\ W\ b,\ a_5\ NW\ b,\ a_6\ N\ b,\ a_7\ SE\ b$ and $a_8\ E\ b$.

In general each multi-tile cardinal direction relation is defined as follows. If $2 \leqslant k \leqslant 9$ then

$a\ R_1{:}\cdots{:}R_k\ b$ iff there exist regions $a_1, \dots, a_k \in REG^*$ such that $a = a_1 \cup \cdots \cup a_k,\ a_1\ R_1\ b,\ a_2\ R_2\ b,\ \dots,$ and $a_k\ R_k\ b$.

The variables $a_1, \dots, a_k$ in any equivalence such as the above are in general in REG^*. For instance, let us consider Fig. 6(a). The lines forming the bounding box of the reference region b divide region $a \in REG$ into two components a_1 and a_2. Clearly a_2 is in REG^* but not in REG. Notice also that for every i, j such that $1 \leqslant i, j \leqslant k$ and $i \neq j$, a_i and a_j have disjoint interiors but may share points in their boundaries.

Each of the above cardinal direction relations can also be defined using set-theoretic notation as *binary relations* consisting of all pairs of regions satisfying the right-hand sides of the "iff" definitions. The reader should keep this in mind throughout the paper; this equivalent way of defining cardinal direction relations will be very useful in Section 6.

The set of basic cardinal direction relations in this model contains 218 elements. We will use $\mathcal{D}$ to denote this set. Relations in $\mathcal{D}$ are jointly exhaustive and pairwise disjoint. Elements of $\mathcal{D}$ can be used to represent *definite information* about cardinal directions, e.g., $a\ N\ b$. An enumeration and a pictorial representation for all relations in $\mathcal{D}$ can be found in [19].

Using the 218 relations of $\mathcal{D}$ as our basis, we can define the *powerset* $2^{\mathcal{D}}$ of $\mathcal{D}$ which contains 2^{218} relations. Elements of $2^{\mathcal{D}}$ are called *cardinal direction relations* and can be used to represent not only definite but also *indefinite information* about cardinal directions, e.g., $a\ \{N,\ W\}\ b$ denotes that region a is north *or* west of region b. [19] considers only a small subset of the disjunctive relations of $2^{\mathcal{D}}$ through a nice pictorial representation called the *direction-relation* matrix.

Definition 2. Let $R \in 2^{\mathcal{D}}$. The *inverse* of relation R, denoted by $inv(R)$, is another cardinal direction relation which satisfies the following. For arbitrary regions $a, b \in REG$, $a\ inv(R)\ b$ holds, iff $b\ R\ a$ holds.

Let us consider two regions a and b and assume that $a\ R\ b$ where R is a basic cardinal direction relation. Then relation $inv(R)$ is not necessarily a basic cardinal direction relation but it can also be a disjunction of basic relations. For instance, if $a\ N\ b$ then it is possible that $b\ SE{:}S{:}SW\ a$ or $b\ SE{:}S\ a$ or $b\ S{:}SW\ a$ or $b\ S\ a$ (see Fig. 7). Therefore

$$inv(N) = \{S{:}SW{:}SE,\ S{:}SW,\ SE{:}S,\ S\}.$$

In other words the relative position of two regions a and b is characterized by the pair (R_1, R_2), where R_1 and R_2 are cardinal directions such that $a\ R_1\ b$ and $b\ R_2\ a$, where R_1 is a disjunct of $inv(R_2)$ and R_2 is a disjunct of $inv(R_1)$. An algorithm for computing the inverse relation is discussed in [40].

In the following sections, we will study the operation of composition for cardinal direction relations. Research has focused on two definitions of composition [5,26]. The first one is the standard existential definition from set theory [5,26,32,38].

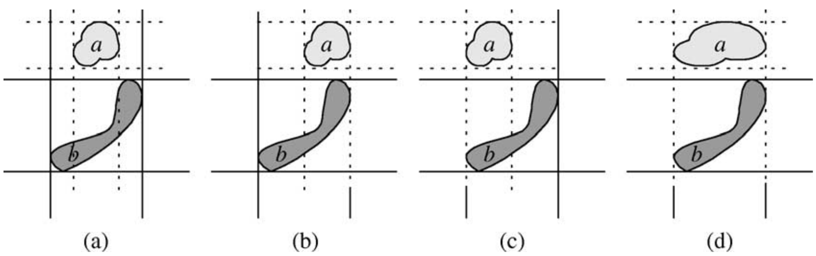

Fig. 7. Members of $inv(N)$.

Definition 3. Let R_1 and R_2 be cardinal direction relations. The *existential composition*[2] of relations R_1 and R_2, denoted by $R_1; R_2$, is another cardinal direction relation from $2^{\mathcal{D}}$ which satisfies the following. For arbitrary regions a and c, $a\ R_1; R_2\ c$ holds if and only if there exists a region b such that $a\ R_1\ b$ and $b\ R_2\ c$ hold.

Definition 3 guarantees that:

(1) If there exist regions a, b and c such that $a\ R_1\ b$ and $b\ R_2\ c$ hold we must have $a\ R_1; R_2\ c$.
(2) If there exist regions a and c such that $a\ R_1; R_2\ c$ holds then there exists a region b such that $a\ R_1\ b$ and $b\ R_2\ c$.

The second definition is as follows [5,26].

Definition 4. Let R_1 and R_2 be cardinal direction relations. The *consistency-based composition* of relations R_1 and R_2, denoted by $R_1 \circ R_2$, is another cardinal direction relation from $2^{\mathcal{D}}$ which satisfies the following. $R_1 \circ R_2$ contains all relations $Q \in \mathcal{D}$ such that there exist regions $a, b, c \in \mathit{REG}$ such that $a\ R_1\ b$, $b\ R_2\ c$ and $a\ Q\ c$ hold.

The consistency-based definition of composition is weaker that the existential definition. Observe that $R_1; R_2 \subseteq R_1 \circ R_2$ holds. The above definitions are very important and have attracted the interest of many researchers since they can be used as a mechanism for inferring new information from existing one [5,11,19,26,36,38].

In this paper, we will first study consistency-based composition for cardinal direction constraints. We will see that this operator is expressible for every pair of cardinal direction relations and can be naturally used as a constraint propagation mechanism. Then, Section 6 discusses existential composition. In contrast with consistency-based composition, existential composition is *not* expressible, using the vocabulary defined in [19], for every pair of cardinal direction relations and thus cannot be used as a constraint propagation mechanism. From this point onwards, unless specifically stated, the term composition refers to consistency-based composition.

Goyal and Egenhofer first studied the composition operation for cardinal direction relations in [19]. Unfortunately, the method presented in [19] does not calculate the correct composition in several cases (see Section 4).

We will address the composition problem one step at a time. So let us first consider, the case of composing a single-tile cardinal direction relation with a basic (single-tile or multi-tile) cardinal direction relation.

[2] The term existential composition is widely used in relevant literature [5,26]. Other researchers prefer the term *extensional composition* [28].

$R_1 \setminus R_2$	N	NE	E	SE	S	SW	W	NW	B
N	N	NE	$\delta(NE, E)$	$\delta(NE, E, SE)$	$\delta(S, N, B)$	$\delta(SW, W, NW)$	$\delta(W, NW)$	NW	$\delta(N, B)$
NE	$\delta(N, NE)$	NE	$\delta(NE, E)$	$\delta(NE, E, SE)$	$\delta(N, NE, E, SE, S, B)$	U_{dir}	$\delta(W, NW, N, NE, E, B)$	$\delta(NW, N, NE)$	$\delta(B, N, NE, E)$
E	$\delta(N, NE)$	NE	E	SE	$\delta(SE, S)$	$\delta(SE, S, SW)$	$\delta(W, E, B)$	$\delta(NW, N, NE)$	$\delta(E, B)$
SE	$\delta(N, NE, E, SE, S, B)$	$\delta(NE, E, SE)$	$\delta(E, SE)$	SE	$\delta(SE, S)$	$\delta(SE, S, SW)$	$\delta(E, SE, S, SW, W, B)$	U_{dir}	$\delta(B, S, E, SE)$
S	$\delta(N, S, B)$	$\delta(NE, E, SE)$	$\delta(E, SE)$	SE	S	SW	$\delta(SW, W)$	$\delta(SW, W, NW)$	$\delta(S, B)$
SW	$\delta(S, SW, W, NW, N, B)$	U_{dir}	$\delta(E, SE, S, SW, W, B)$	$\delta(SE, S, SW)$	$\delta(S, SW)$	SW	$\delta(SW, W)$	$\delta(SW, W, NW)$	$\delta(B, S, SW, W)$
W	$\delta(NW, N)$	$\delta(NW, N, NE)$	$\delta(E, W, B)$	$\delta(SE, S, SW)$	$\delta(S, SW)$	SW	W	NW	$\delta(W, B)$
NW	$\delta(NW, N)$	$\delta(NW, N, NE)$	$\delta(W, NW, N, NE, E, B)$	U_{dir}	$\delta(S, SW, W, NW, N, B)$	$\delta(SW, W, NW)$	$\delta(W, NW)$	NW	$\delta(B, W, NW, N)$
B	N	NE	E	SE	S	SW	W	NW	B

Fig. 8. The composition $R_1 \circ R_2$ of single-tile relations R_1 and R_2.

4. Composing a single-tile relation with a basic relation

In Fig. 8 we show the composition table for single-tile cardinal direction relations [19]. The table uses the function symbol δ as a shortcut. For arbitrary single-tile cardinal direction relations $R_1, \ldots, R_k$, the notation $\delta(R_1, \ldots, R_k)$ is a shortcut for the disjunctive relations in $\mathcal{D}^*$ that can be constructed by combining single-tile relations $R_1, \ldots, R_k$. For instance, $\delta(SW, W, NW)$ stands for the disjunctive relation:

$$\{SW,\ W,\ NW,\ SW{:}W,\ W{:}NW,\ SW{:}W{:}NW\}.$$

Moreover, we define:

$$\begin{aligned}&\delta\big(\delta(R_{11}, \ldots, R_{1k_1}), \delta(R_{21}, \ldots, R_{2k_2}), \ldots, \delta(R_{m1}, \ldots, R_{mk_m})\big)\\ &\quad = \delta(R_{11}, \ldots, R_{1k_1}, R_{21}, \ldots, R_{2k_2}, \ldots, R_{m1}, \ldots, R_{mk_m}).\end{aligned}$$

Application of the operator δ as it has been defined, suffices for our needs in this paper.

As usual U_{dir} stands for the universal cardinal direction relation. The correctness of the composition table of Fig. 8 can easily be verified using the definitions of Section 3 and the definition of composition (Definition 4).

We now turn our attention to the problem of the composition of a single-tile cardinal direction relation with some "well-behaved" basic cardinal direction relations. We will need the following definition.

Definition 5. A basic cardinal direction relation R is called *rectangular* iff there exist two rectangles (with sides parallel to the x- and y-axes) a and b such that $a\ R\ b$ is satisfied; otherwise it is called *non-rectangular*.

Example 2. All single-tile relations are rectangular. Relations $B{:}N$ and $B{:}S{:}SW{:}W$ are rectangular while relations $B{:}S{:}SW$ and $B{:}S{:}N{:}SE$ are non-rectangular.

The set of rectangular cardinal direction relations contains the following 36 relations:

$$\{B,\ S,\ SW,\ W,\ NW,\ N,\ NE,\ E,\ SE, S{:}SW,\ B{:}W,\ NW{:}N,\ N{:}NE,\ B{:}E,\ S{:}SE,\ SW{:}W,\ B{:}S,\ E{:}SE,\ W{:}NW, B{:}N,\ NE{:}E,\ S{:}SW{:}SE,\ NW{:}N{:}NE,\ B{:}W{:}E,\ B{:}S{:}N, SW{:}W{:}NW, NE{:}E{:}SE,\ B{:}S{:}SW{:}W,\ B{:}W{:}NW{:}N,\ B{:}S{:}E{:}SE, B{:}N{:}NE{:}E,\ B{:}S{:}SW{:}W{:}NW{:}N,\ B{:}S{:}N{:}NE{:}E{:}SE,\ B{:}S{:}SW{:}W{:}E{:}SE,\ B{:}W{:}NW{:}N{:}NE{:}E,\ B{:}S{:}SW{:}W{:}NW{:}N{:}NE{:}E{:}SE\}.$$

Lemma 1. *Let R_1 be a single-tile and $R_{21}{:}\cdots{:}R_{2l}$ be a basic cardinal direction relation (single-tile or multi-tile). The composition of R_1 with $R_{21}{:}\cdots{:}R_{2l}$ can be computed using the formula*

$$R_1 \circ (R_{21}{:}\cdots{:}R_{2l}) = \delta(R_1 \circ R_{21}, \ldots, R_1 \circ R_{2l}) \tag{1}$$

in any of the following cases:

(i) *if $R_1 \in \{W, E\}$ and $R_{21}{:}\cdots{:}R_{2l} \in \{B, S, SW, W, NW, N, NE, E, SE, W{:}NW, B{:}N, NE{:}E, SW{:}W, B{:}S, E{:}SE, NW{:}W{:}SW, N{:}B{:}S, SE{:}E{:}NE\}$.*
(ii) *if $R_1 \in \{S, N\}$ and $R_{21}{:}\cdots{:}R_{2l} \in \{B, S, SW, W, NW, N, NE, E, SE, NW{:}N, N{:}NE, B{:}W, B{:}E, SW{:}S, S{:}SE, NW{:}N{:}NE, W{:}B{:}E, S{:}SW{:}SE\}$.*
(iii) *if $R_1 \in \{NW, SW, NE, SE\}$ and $R_{21}{:}\cdots{:}R_{2l}$ is any single-tile cardinal direction relation in $\mathcal{D}$.*
(iv) *if $R_1 = B$ and $R_{21}{:}\cdots{:}R_{2l}$ is any rectangular cardinal direction relation in $\mathcal{D}$.*

Proof. For case (i), let us first assume that $R_1 = W$. We can prove, by a simple case analysis, that for any $R_{21}{:}\cdots{:}R_{2l} \in \{B, S, SW, W, NW, N, NE, E, SE, W{:}NW, B{:}N, NE{:}E, SW{:}W, B{:}S, E{:}SE, NW{:}W{:}SW, N{:}B{:}S, SE{:}E{:}NE\}$ Eq. (1) holds. For instance, if $R_{21}{:}\cdots{:}R_{2l} = SW{:}W$, Eq. (1) gives:

$$W \circ SW{:}W = \delta(W \circ SW, W \circ W) = \delta(SW, W) = \{SW,\ W,\ SW{:}W\}.$$

This result can be easily verified (see also Fig. 9). The case where $R_1 = E$ is symmetric.

Case (ii) is symmetric to case (i). Finally, cases (iii) and (iv) can trivially be verified. □

We now turn our attention to the composition of a single-tile with an arbitrary basic cardinal direction relation. Goyal and Egenhofer [19] apply Eq. (1) of Lemma 1 to *all* pairs of single-tile and basic cardinal direction relations. Unfortunately, this is not correct. For instance, according to the composition method of [19], one would have $W \circ NW{:}N = \{NW, N, NW{:}N\}$ but the correct composition is $W \circ NW{:}N = \{NW\}$ (as we can see in Fig. 10).

Let R_1 be a single-tile and R_2 be a basic cardinal direction relation (single-tile or multi-tile). Intuitively, to compute the composition $R_1 \circ R_2$, we transform R_2 into a relation Q

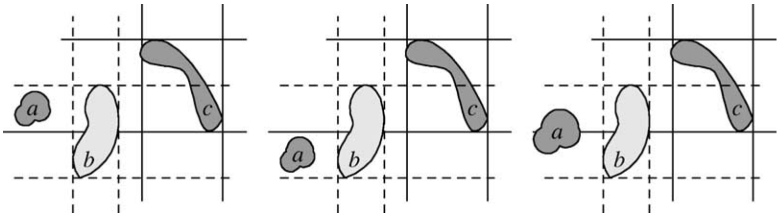

Fig. 9. Composing W with $SW{:}W$.

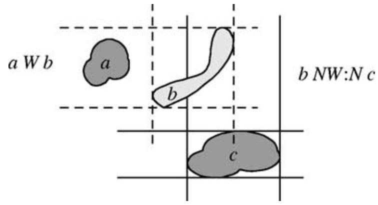

Fig. 10. Composing W with $NW{:}N$.

such that $R_1 \circ R_2 = R_1 \circ Q$ and $R_1 \circ Q$ can be calculated using Eq. (1). This transformation is achieved using the operators $\mathcal{B}r$ and $\mathcal{M}ost$ defined below. Before we present our result (Theorem 1), we will first need a few definitions and lemmas.

Definition 6. Let $R_1 = R_{11}{:}\cdots{:}R_{1k}$ and $R_2 = R_{21}{:}\cdots{:}R_{2l}$ be two cardinal direction relations. R_1 *includes* R_2 iff $\{R_{21}, \ldots, R_{2l}\} \subseteq \{R_{11}, \ldots, R_{1k}\}$ holds.

Example 3. The basic cardinal direction relation $B{:}S{:}SW{:}W$ includes relation $B{:}S{:}SW$.

Definition 7. Let R be a basic cardinal direction relation. The *bounding relation* of R, denoted by $\mathcal{B}r(R)$ is the smallest rectangular relation (with respect to the number of tiles) that includes R.

Example 4. The bounding relation of the basic cardinal direction relation $B{:}S{:}SW$ is relation $B{:}S{:}SW{:}W$.

The notion of direction can be also applied to the tiles of a relation. Thus, B is west of E and northeast of SE. Moreover, given a relation $R = R_1{:}\cdots{:}R_n$, R_i is a westernmost tile of R if it is west of every other tile of R. For instance SW is a westernmost tile of $B{:}S{:}SW{:}W$. Similarly, we define southwesternmost, southernmost, southeasternmost, easternmost, northeasternmost, northernmost and northwesternmost tiles of a relation.

Definition 8. Let R be a rectangular cardinal direction relation. We will denote the rectangular relation formed by the westernmost tiles of a relation R by $\mathcal{M}ost(W, R)$. Similarly, we can define the rectangular relations $\mathcal{M}ost(S, R)$, $\mathcal{M}ost(N, R)$ and $\mathcal{M}ost(E, R)$. Moreover, we will denote the single-tile relation formed by the southwesternmost tiles of a relation R by $\mathcal{M}ost(SW, R)$. Similarly, we can define the single-tile relations $\mathcal{M}ost(SE, R)$, $\mathcal{M}ost(NW, R)$ and $\mathcal{M}ost(NE, R)$. Finally, as a special case, we define $\mathcal{M}ost(B, R) = R$.

Example 5. Let us consider the rectangular relation $B{:}S{:}SW{:}W$. Then according to Definition 8 we have:

$$\mathcal{M}ost(W, B{:}S{:}SW{:}W) = SW{:}W, \qquad \mathcal{M}ost(SE, B{:}S{:}SW{:}W) = S,$$
$$\mathcal{M}ost(S, B{:}S{:}SW{:}W) = S{:}SW, \qquad \mathcal{M}ost(SW, B{:}S{:}SW{:}W) = SW,$$
$$\mathcal{M}ost(E, B{:}S{:}SW{:}W) = B{:}S, \qquad \mathcal{M}ost(NW, B{:}S{:}SW{:}W) = W,$$
$$\mathcal{M}ost(N, B{:}S{:}SW{:}W) = B{:}W, \qquad \mathcal{M}ost(NE, B{:}S{:}SW{:}W) = B,$$
$$\mathcal{M}ost(B, B{:}S{:}SW{:}W) = B{:}S{:}SW{:}W.$$

We will see later, in Theorem 1, that operator $\mathcal{M}ost$ is crucial for the correct composition of a single-tile with a basic cardinal direction relation. The following lemma expresses an important property of operator $\mathcal{M}ost$.

Lemma 2. *Let R_1 be a single-tile and R_2 be a rectangular cardinal direction relation. Assume that relation $\mathcal{M}ost(R_1, R_2)$ is $Q_1{:}\cdots{:}Q_t$. Then, the composition of R_1 with $\mathcal{M}ost(R_1, R_2)$ can be computed using the formula*:

$$R_1 \circ \mathcal{M}ost(R_1, R_2) = \delta(R_1 \circ Q_1, \ldots, R_1 \circ Q_t). \tag{2}$$

Proof. The proof is by case analysis. Let us first assume that $R_1 = W$. For any $R_2 \in \mathcal{D}$, the relation $\mathcal{M}ost(W, R_2)$ will be in one of the following relations:

- Single-tile relations: B, S, SW, W, NW, N, NE, E, SE.
- Two-tile relations: $W{:}NW$, $B{:}N$, $NE{:}E$, $SW{:}W$, $B{:}S$, $E{:}SE$.
- Three-tile relations: $NW{:}W{:}SW$, $N{:}B{:}S$, $SE{:}E{:}NE$.

We can now see that for all the above cases the conditions of Lemma 1 hold, so the composition of W with $\mathcal{M}ost(W, R_2)$ can be calculated using Eq. (1) of Lemma 1. This is exactly what Eq. (2) does.

Similarly, we can prove that Eq. (2) holds for any other single-tile cardinal direction relation $R_1 \in \{B, S, SW, NW, N, NE, E, SE\}$. In all cases Lemma 1 is used. □

The following is one more useful lemma.

Lemma 3. *Let R_1 be a single-tile relation and R_2 be a basic cardinal direction relation. The following implications hold*:

(i) $(\forall a, b, c \in REG)(a\ R_1\ b \wedge b\ R_2\ c \rightarrow$
$(\exists d \in REG)(a\ R_1\ d \wedge d\ \mathcal{B}r(R_2)\ c))$.

(ii) $(\forall a, b, c \in REG)(a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c \rightarrow$
$(\exists d \in REG)(a\ R_1\ d \wedge d\ R_2\ c))$.

(iii) $(\forall a, b, c \in REG)(a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c \rightarrow$

$$(\exists d \in REG)(a\ R_1\ d \wedge d\ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c)).$$

(iv) $(\forall a, b, c \in REG)(a\ R_1\ b \wedge b\ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c \rightarrow$

$$(\exists d \in REG)(a\ R_1\ d \wedge d\ \mathcal{B}r(R_2)\ c)).$$

Proof. (i) This is easy to see. If $a\ R_1\ b \wedge b\ R_2\ c$ holds, then $a\ R_1\ mbb(b) \wedge mbb(b)\ \mathcal{B}r(R_2)\ c$ also holds.

For instance, if $a\ W\ b \wedge b\ B{:}S{:}SW\ c$ holds (Fig. 11(a)), then

$$a\ W\ mbb(b) \wedge mbb(b)\ B{:}S{:}SW{:}W\ c$$

also holds, where $B{:}S{:}SW{:}W = \mathcal{B}r(B{:}S{:}SW)$.

(ii) Assume that R_2 is $R_{21}{:}\cdots{:}R_{2k}$ where $R_{21}, \ldots, R_{2k}$ are single-tile cardinal direction relations. Let a, b, c be regions in REG and $a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c$ holds. Consider the tiles $R_{21}(c), \ldots, R_{2k}(c)$ and form the region $d_0 = mbb(b) \cap (R_{21}(c) \cup \cdots \cup R_{2k}(c))$. Then, $a\ R_1\ d_0$ holds because d_0 and b have the same minimum bounding box. Also $d_0\ R_2\ c$ holds by the construction of d_0.

For instance, if $a\ W\ b \wedge b\ \mathcal{B}r(B{:}S{:}SW)\ c$ holds (Fig. 11(b)), then $a\ W\ d_0 \wedge d_0\ B{:}S{:}SW\ c$ also holds, where $d_0 = mbb(b) \cap (B(c) \cup S(c) \cup SW(c))$ (the light grey area of Fig. 11(b)).

(iii) The proof is by case analysis. Let us first assume that $R_1 = W$. Assume also that $\mathcal{M}ost(W, \mathcal{B}r(R_2))$ is $Q_1{:}\cdots{:}Q_t$, where $Q_1, \ldots, Q_t$ are single-tile cardinal direction relations. Let a, b, c be regions in REG and $a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c$ holds. Let us form region $d_0 = mbb(b) \cap (Q_1(c) \cup \cdots \cup Q_t(c))$. Then, $a\ W\ d_0 \wedge d_0\ \mathcal{M}ost(W, \mathcal{B}r(R_2))\ c$ holds by the definition of $\mathcal{M}ost(W, \mathcal{B}r(R_2))$ (Definition 8) and the construction of d_0. Thus, the implication holds.

For instance, if $a\ W\ b \wedge b\ \mathcal{B}r(B{:}S{:}SW)\ c$ holds (Fig. 11(c)), then

$$a\ W\ d_0 \wedge d_0\ \mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW))\ c$$

also holds, where $SW{:}W = \mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW))$ and $d_0 = mbb(b) \cap (SW(c) \cup W(c))$ (the light grey area of Fig. 11(b)).

Similarly, the lemma holds for any other basic cardinal direction relation $R_1 \in \{B, S, SW, NW, N, NE, E, SE\}$.

(iv) This proof is by case analysis. Let us first assume that $R_1 = W$. Assume also that $\mathcal{B}r(R_2)$ is $Q_1{:}\cdots{:}Q_t$, where $Q_1, \ldots, Q_t$ are single-tile cardinal direction relations. Let

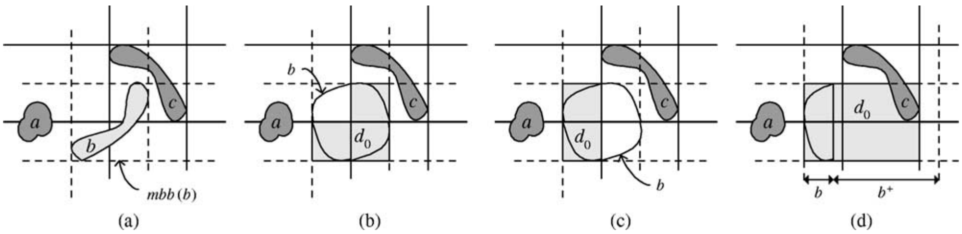

Fig. 11. Proving Lemma 3.

a, b, c be regions in REG and $a\ R_1\ b \wedge b\ \mathcal{M}ost(W, \mathcal{B}r(R_2))\ c$ holds. Let us form a box $b^+ \in REG$ such that:

$$sup_y(b^+) = sup_y(b), \qquad inf_y(b^+) = inf_y(b),$$
$$inf_x(b^+) = sup_x(b), \qquad sup_x(b^+) > sup_x(c)$$

and region $d_0 = (mbb(b) \cup b^+) \cap (Q_1(c) \cup \cdots \cup Q_t(c))$. Then, $a\ W\ d_0 \wedge d_0\ \mathcal{B}r(R_2)\ c$ holds by the construction of d_0. Thus, the implication holds.

For instance, if $a\ W\ b \wedge b\ \mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW))\ c$ holds (Fig. 11(d)), then $a\ W\ d_0 \wedge d_0\ \mathcal{B}r(B{:}S{:}SW)\ c$ also holds, where $B{:}S{:}SW{:}W = \mathcal{B}r(B{:}S{:}SW)$ and $d_0 = (mbb(b) \cup b^+) \cap (B(c) \cup S(c) \cup SW(c) \cup W(c))$ (the light grey area of Fig. 11(d)).

Similarly, the lemma holds for any other basic cardinal direction relation $R_1 \in \{B, S, SW, NW, N, NE, E, SE\}$. The proofs for these cases only vary in the way they construct b^+. □

Now, after all the necessary definitions and lemmas, we can present our result.

Theorem 1. *Let R_1 be a single-tile cardinal direction relation and R_2 be a basic cardinal direction relation. Then*

$$R_1 \circ R_2 = R_1 \circ \mathcal{M}ost\big(R_1, \mathcal{B}r(R_2)\big). \tag{3}$$

Proof. Let $Q \in R_1 \circ R_2$. Then, according to Definition 4 there exist regions $a, b, c \in REG$ such that:

$$a\ R_1\ b \wedge b\ R_2\ c \wedge a\ Q\ c.$$

Then, according to Lemma 3(i) there exists a region $e \in REG$ such that:

$$a\ R_1\ e \wedge e\ \mathcal{B}r(R_2)\ c \wedge a\ Q\ c.$$

Then, according to Lemma 3(iii) there exists a region $g \in REG$ such that:

$$a\ R_1\ g \wedge g\ \mathcal{M}ost\big(R_1, \mathcal{B}r(R_2)\big)\ c \wedge a\ Q\ c.$$

Therefore, we have $Q \in R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))$.

Conversely, let $Q \in R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))$. Then, according to Definition 4 there exist regions $a, b, c \in REG$ such that:

$$a\ R_1\ b \wedge b\ \mathcal{M}ost\big(R_1, \mathcal{B}r(R_2)\big)\ c \wedge a\ Q\ c.$$

Then, according to Lemma 3(iv) there exists a region $e \in REG$ such that:

$$a\ R_1\ e \wedge e\ \mathcal{B}r(R_2)\ c \wedge a\ Q\ c.$$

Then, according to Lemma 3(ii) there exists a region $g \in REG$ such that:

$$a\ R_1\ g \wedge g\ R_2\ c \wedge a\ Q\ c.$$

Therefore, we have $Q \in R_1 \circ R_2$ and thus the theorem holds. □

The above theorem give us a method to compute the composition $R_1 \circ R_2$ of a single-tile cardinal direction relation R_1 with a basic cardinal direction relation R_2. First we have to

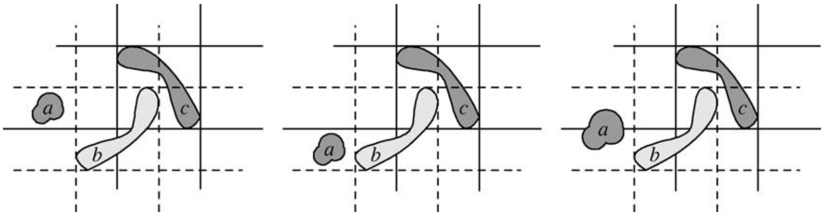

Fig. 12. Composing a single-tile with a basic cardinal direction relation.

calculate the relation $\mathcal{M}ost(R_1, \mathcal{B}r(R_2))$. Then we use Lemma 2 and the table of Fig. 8 to compute $R_1 \circ R_2$.

We illustrate the above procedure in the following example.

Example 6. Let $R_1 = W$ be a single-tile and $R_2 = B{:}S{:}SW$ be a basic cardinal direction relation. Then

$$\mathcal{M}ost\big(W, \mathcal{B}r(B{:}S{:}SW)\big) = SW{:}W.$$

Thus, the composition $W \circ B{:}S{:}SW{:}W$ can be calculated using Theorem 1 as follows:

$$W \circ B{:}S{:}SW = W \circ SW{:}W.$$

Using Lemma 2 we have:

$$W \circ B{:}S{:}SW = \delta(W \circ SW, W \circ W).$$

Using the table of Fig. 8 we equivalently have:

$$W \circ B{:}S{:}SW = \delta(SW, W).$$

Finally, expanding operator δ we have:

$$W \circ B{:}S{:}SW = \{SW,\ W,\ SW{:}W\}.$$

The above equation can be easily verified (see also Fig. 12).

5. Composing basic cardinal direction relations

In this section we will study the composition of basic cardinal direction relations. We will need the following definition.

Definition 9. Let R_1 and R_2 be two basic cardinal direction relations. The *tile-union* of R_1 and R_2, denoted by *tile-union*(R_1, R_2), is the basic cardinal direction relation that consists of all the tiles in R_1 and R_2.

For instance, if $R_1 = B{:}S{:}SW$ and $R_2 = S{:}SW{:}W$ then

$$\textit{tile-union}(R_1, R_2) = B{:}S{:}SW{:}W.$$

Note that the result of *tile-union* is not always a valid cardinal direction relation. For instance, if $R_1 = W$ and $R_2 = E$ then *tile-union*$(R_1, R_2) = W{:}E \notin \mathcal{D}$.

Before we present our result, let us consider the following lemma.

Lemma 4. *Let R_1 be a single-tile relation and R_2 be a basic cardinal direction relation. If $Q \in R_1 \circ R_2$ then the following implication holds*:

$$(\forall b, c \in REG)(b\ R_2\ c \rightarrow (\exists a \in REG)(a\ R_1\ b \wedge a\ Q\ c)).$$

Proof. Let us assume that $R_1 = W$. According to Theorem 1 we have

$$W \circ R_2 = W \circ \mathcal{M}ost(W, \mathcal{B}r(R_2)).$$

Tables 1 and 2 show all the possible values of $\mathcal{B}r(R_2)$ and $\mathcal{M}ost(W, \mathcal{B}r(R_2))$.

Thus, the possible values of expression $W \circ \mathcal{M}ost(W, \mathcal{B}r(R_2))$ are given by Tables 3 and 4.

Assume that $Q \in W \circ R_2$ is of the form $Q_1{:}\cdots{:}Q_t$, where $Q_1, \ldots, Q_t$ are single-tile cardinal direction relations. Let b, c be regions in REG such that $b\ R_2\ c$ holds. Let us first form a box $a^+ \in REG$ such that:

$$sup_y(a^+) > max\{sup_y(b), sup_y(c)\}, \qquad inf_y(a^+) < min\{inf_y(b), inf_y(c)\},$$
$$sup_x(a^+) > max\{sup_x(b), sup_x(c)\}, \qquad inf_x(a^+) < min\{inf_x(b), inf_x(c)\}$$

and region $a_0 = a^+ \cap W(b) \cap (Q_1(c) \cup \cdots \cup Q_t(c))$. Since $Q \in W \circ R_2$ holds, region a_0 is non-empty. Moreover, $a_0\ W\ b \wedge a_0\ Q\ c$ holds by the construction of a_0. Thus, the implication holds.

For instance, let $R_2 = SW{:}W$ (this case corresponds to the first entry of Table 4) and $Q = SW{:}W \in (W \circ SW{:}W)$ hold. If $b\ SW{:}W\ c$ holds (Fig. 13) then $a_0\ W\ b \wedge a_0\ SW{:}W\ c$ also holds, where $a_0 = a^+ \cap W(b) \cap (SW(c) \cup W(c))$ (the light grey area of Fig. 13).

Table 1

$\mathcal{B}r(R_2)$	$\mathcal{M}ost(W, \mathcal{B}r(R_2))$
B	B
S	S
SW	SW
W	W
NW	NW
N	N
NE	NE
E	E
SE	SE
$S{:}SW$	SW
$B{:}W$	W
$NW{:}N$	NW
$N{:}NE$	N
$B{:}E$	B
$S{:}SE$	S
$SW{:}W$	$SW{:}W$
$B{:}S$	$B{:}S$
$E{:}SE$	$E{:}SE$

Table 2

$\mathcal{B}r(R_2)$	$\mathcal{M}ost(W, \mathcal{B}r(R_2))$
$W{:}NW$	$W{:}NW$
$B{:}N$	$B{:}N$
$NE{:}E$	$NE{:}E$
$S{:}SW{:}SE$	SW
$NW{:}N{:}NE$	NW
$B{:}W{:}E$	W
$B{:}S{:}N$	$B{:}S{:}N$
$SW{:}W{:}NW$	$SW{:}W{:}NW$
$NE{:}E{:}SE$	$NE{:}E{:}SE$
$B{:}S{:}SW{:}W$	$SW{:}W$
$B{:}W{:}NW{:}N$	$W{:}NW$
$B{:}S{:}E{:}SE$	$B{:}S$
$B{:}N{:}NE{:}E$	$B{:}N$
$B{:}S{:}SW{:}W{:}NW{:}N$	$SW{:}W{:}NW$
$B{:}S{:}N{:}NE{:}E{:}SE$	$B{:}S{:}N$
$B{:}S{:}SW{:}W{:}E{:}SE$	$SW{:}W$
$B{:}W{:}NW{:}N{:}NE{:}E$	$W{:}NW$
$B{:}S{:}SW{:}W{:}NW{:}N{:}NE{:}E{:}SE$	$SW{:}W{:}NW$

Table 3

$W \circ \mathcal{M}ost(W, \mathcal{B}r(R_2))$
$W \circ B$
$W \circ S$
$W \circ SW$
$W \circ W$
$W \circ NW$
$W \circ N$
$W \circ NE$
$W \circ E$
$W \circ SE$

Table 4

$W \circ \mathcal{M}ost(W, \mathcal{B}r(R_2))$
$W \circ SW{:}W$
$W \circ B{:}S$
$W \circ E{:}SE$
$W \circ W{:}NW$
$W \circ B{:}N$
$W \circ NE{:}E$
$W \circ B{:}S{:}N$
$W \circ SW{:}W{:}NW$
$W \circ NE{:}E{:}SE$

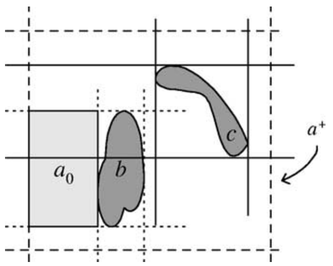

Fig. 13. Proving Lemma 4.

The cases where $R_1 = N$, $R_1 = E$ and $R_1 = S$ are symmetric while the cases where $R_1 = NW$, $R_1 = NE$, $R_1 = SE$, $R_1 = SW$ and $R_1 = B$ can be proved similarly. □

Now, we are ready to present our result.

Theorem 2. *Let $R_1 = R_{11}{:}\cdots{:}R_{1k}$ and R_2 be basic cardinal direction relations, where $R_{11}, \ldots, R_{1k}$ are single-tile cardinal direction relations. Then*

$$R_1 \circ R_2 = \big\{Q \in \mathcal{D}\colon\ (\exists s_1, \ldots, s_k)\ (Q = \textit{tile-union}(s_1, \ldots, s_k)\ \wedge \\ s_1 \in R_{11} \circ R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ R_2)\big\}.$$

Proof. The composition $R_1 \circ R_2$ is defined (Definition 4) as follows:

$$R_1 \circ R_2 = \big\{Q \in \mathcal{D}\colon\ (\exists a, b, c)(a\ R_1\ b \wedge b\ R_2\ c \wedge a\ Q\ c)\big\}.$$

Let $\mathcal{C}_{R_1,R_2} = \{Q \in \mathcal{D}\colon\ Q = \textit{tile-union}(s_1, \ldots, s_k) \wedge s_1 \in R_{11} \circ R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ R_2\}$. We will prove the following equation:

$$R_1 \circ R_2 = \mathcal{C}_{R_1,R_2}. \tag{4}$$

We distinguish two cases.

Case 1 assumes that $k = 1$. Then $R_1 = R_{11}$, i.e., R_1 is a single-tile cardinal direction relation and the result is trivial.

Case 2 assumes that $k > 1$. Let $Q \in R_1 \circ R_2$. Then, according to Definition 4 there exist regions $a, b, c \in \textit{REG}$ such that:

$$a\ R_1\ b \wedge b\ R_2\ c \wedge a\ Q\ c.$$

Since $a\ R_1\ b$ holds, there exist regions $a_1, \ldots, a_k \in \textit{REG}$ such that $a = a_1 \cup \cdots \cup a_k$ and $a_1\ R_{11}\ b \wedge \cdots \wedge a_k\ R_{1k}\ b$ hold. Therefore, we have:

$$(a_1\ R_{11}\ b \wedge \cdots \wedge a_k\ R_{1k}\ b) \wedge b\ R_2\ c \wedge a\ Q\ c.$$

Moreover, since $a = a_1 \cup \cdots \cup a_k$ and $a\ Q\ c$ hold, there exist basic cardinal direction relations $Q_1, \ldots, Q_k$ such that $Q = \textit{tile-union}(Q_1, \ldots, Q_k)$ and $a_1\ Q_1\ c \wedge \cdots \wedge a_k\ Q_k\ c$ hold. Therefore, from the above formula, we have:

$$(a_1\ R_{11}\ b \wedge \cdots \wedge a_k\ R_{1k}\ b) \wedge b\ R_2\ c \wedge (a_1\ Q_1\ c \wedge \cdots \wedge a_k\ R_k\ c).$$

Equivalently, we have:

$$(a_1\ R_{11}\ b \wedge b\ R_2\ c \wedge a_1\ Q_1\ c) \wedge \cdots \wedge (a_k\ R_{1k}\ b \wedge b\ R_2\ c \wedge a_k\ Q_k\ c)$$

thus $Q_1 \in R_{11} \circ R_2, \ldots, Q_k \in R_{1k} \circ R_2$ and since $Q = \textit{tile-union}(Q_1, \ldots, Q_k)$ holds, we have $Q \in \mathcal{C}_{R_1,R_2}$.

Conversely, let $Q \in \mathcal{C}_{R_1,R_2}$. Then, according to the definition of $\mathcal{C}_{R_1,R_2}$ there exist $Q_1 \in R_{11} \circ R_2, \ldots, Q_k \in R_{1k} \circ R_2$ such that $Q = \textit{tile-union}(Q_1, \ldots, Q_k)$ holds. Thus, there exist regions $a_1, \ldots, a_k,\ b_1, \ldots, b_k,\ c_1, \ldots, c_k$ such that the following formula holds.

$$(a_1\ R_{11}\ b_1 \wedge b_1\ R_2\ c_1 \wedge a_1\ Q_1\ c_1) \wedge \cdots \wedge (a_k\ R_{1k}\ b_k \wedge b_k\ R_2\ c_k \wedge a_k\ Q_k\ c_k).$$

Let us now consider two regions b^0 and c^0 in *REG* such that $b^0\ R_2\ c^0$. Then, according to Lemma 4, there exist regions $e_1, \ldots, e_k$ such that the following formula holds.

$$\big(e_1\ R_{11}\ b^0 \wedge b^0\ R_2\ c^0 \wedge e_1\ Q_1\ c^0\big) \wedge \cdots \wedge \big(e_k\ R_{1k}\ b^0 \wedge b^0\ R_2\ c^0 \wedge e_k\ Q_k\ c^0\big).$$

Let us now form region $a^0 = e_1 \cup \cdots \cup e_k$. Then, since $R_1 = R_{11}{:}\cdots{:}R_{1k}$ and $Q = \textit{tile-union}(Q_1, \ldots, Q_k)$ hold, we have:

$$a^0\ R_1\ b^0 \wedge b^0\ R_2\ c^0 \wedge a^0\ Q\ c^0.$$

Therefore, we have $Q \in R_1 \circ R_2$ and thus the theorem holds. □

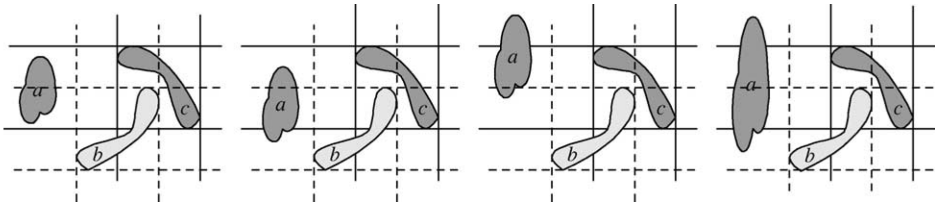

Fig. 14. Composing basic cardinal direction relations.

Using Theorem 2 we can easily derive Algorithm COMPOSE that computes the composition R_3 of basic cardinal direction relations R_1 and R_2. Assume that R_1 is $R_{11}:\cdots:R_{1k}$, where $R_{11},\ldots,R_{1k}$ are single-tile cardinal direction relations. Algorithm COMPOSE proceeds as follows. Initially, the algorithm calculates relations S_i, $1 \leqslant i \leqslant k$, as the composition of the single-tile relation R_{1i} with the basic cardinal direction relation R_2 (as in Section 4). Subsequently, Algorithm COMPOSE forms relations by taking the *tile-union* of a single-tile cardinal direction relation s_i, from every cardinal direction relation S_i $(1 \leqslant i \leqslant k)$. Finally, the algorithm checks whether the result of the union corresponds to a valid cardinal direction relation in $\mathcal{D}$. If it does then this relation is added to the result R_3; otherwise it is discarded.

We have implemented Algorithm COMPOSE and generated the compositions $R_1 \circ R_2$ for every pair of cardinal direction relations R_1 and R_2. The results and the code are available from the authors.

The following is an example of Algorithm COMPOSE in operation.

Example 7. Assume that we want to calculate the composition of basic cardinal direction relations $W{:}NW$ and $B{:}S{:}SW$ (Fig. 14). We have:

$$S_1 = W \circ B{:}S{:}SW = \delta(W \circ SW, W \circ SW) = \delta(W, SW) = \{SW, W, SW{:}W\}$$
$$S_2 = NW \circ B{:}S{:}SW = NW \circ W = \{W, NW, W{:}NW\}.$$

Now we construct all relations formed by the union of one relation from S_1 and one relation from S_2. These relations are: $SW{:}W$, $SW{:}NW$, $SW{:}W{:}NW$, W, $W{:}NW$, $W{:}NW$, $SW{:}W$, $SW{:}W{:}NW$ and $SW{:}W{:}NW$. From the above relations only the following are valid cardinal direction relations:

$$SW{:}W,\ SW{:}W{:}NW,\ W,\ W{:}NW.$$

Therefore, we have:

$$W{:}NW \circ B{:}S{:}SW = \{W,\ SW{:}W,\ W{:}NW,\ SW{:}W{:}NW\}.$$

The above equation can be easily verified (see also Fig. 14).

6. Existential composition of cardinal direction relations

Let us now leave the consistency based definition of composition and consider the standard notion of existential composition from set theory (Definition 3). For this case

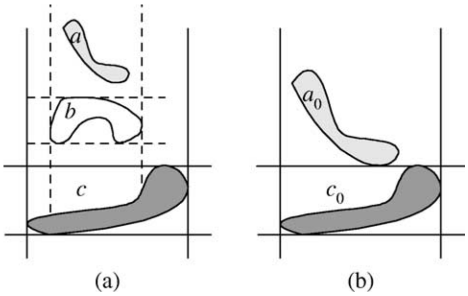

Fig. 15. Illustration of Example 8.

we have the following result: the language of cardinal direction constraints (as defined in Section 3) *is not expressive enough* to capture the binary relation which is the result of the existential composition of cardinal direction relations. This is illustrated by the following example.

Example 8. Consider region variables a, b, c and cardinal direction constraints $a\ N\ b$ and $b\ N\ c$. The only cardinal direction constraint implied by these two constraints is $a\ N\ c$ (see Fig. 15(a)). This is captured by the fact that $N \circ N = N$ (see Fig. 8). Someone would be tempted to conclude that $(N; N) = N$ also holds. If this equality was correct then for each pair of regions a_0 and c_0 such that $a_0\ N\ c_0$, there exists a region $b_0 \in REG$ such that $a_0\ N\ b_0$ and $b_0\ N\ c_0$. However, Fig. 15(b) shows two such regions a_0 and c_0 such that $a_0\ N\ c_0$ and it is impossible to find a region $b_0 \in REG$ such that $a_0\ N\ b_0$ and $b_0\ N\ c_0$.

If we consider Fig. 15(a) carefully, we will notice that the semantics of existential composition imply the following constraints on a and c:

(1) region a lies completely on the north tile of c (i.e., $a\ N\ c$ holds), and
(2) the minimum bounding boxes of regions a and c do not touch.

Intuitively, the second constraint is not expressible in the language of cardinal direction relations presented in Section 3.

It is an open question to define an appropriate set of relations that could be used to augment the constraint language of Section 3 so that the constraints needed to define the result of existential composition are expressible.

7. Discussion

Let us summarize what we have achieved so far. In Sections 4 and 5, we have developed a method to compute the consistency-based composition of basic cardinal direction relations. Moreover, in Section 6, we have seen intuitively that existential composition is not expressive in the language of cardinal direction constraints. Similar results are common in Relation Algebras [42,43] and qualitative spatial reasoning [10,

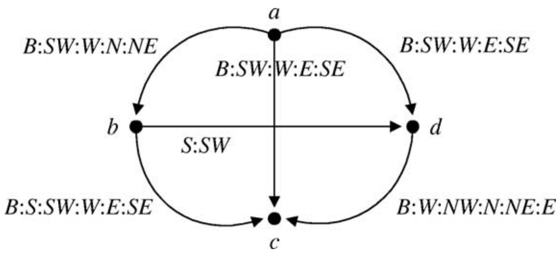

Fig. 16. Illustration of Example 9.

28]. For instance, consider the topological relation set of RCC8 [35]. Even for this case, existential composition cannot be expressed using only RCC8 relations (except under certain restricted interpretations – see [28] for a detailed discussion).

Still, it is important to point out that the intuitive non-expressibility result of Example 8 should not deter spatial database practitioners who would like to consider adding the cardinal direction relations described in this paper to their system. The discussion of the introduction (i.e., using the inferences of a composition table for spatial relations in order to prune the search space during optimization of certain queries) still applies but now one has to be careful to say that he is using a constraint propagation mechanism which is formed by consistency-based composition and not existential set-theoretic composition!

Notice that we cannot use consistency-based composition to decide the consistency of a set of basic cardinal direction constraints. This is demonstrated by the following example.

Example 9. Let C be the following set of basic cardinal direction constraints (Fig. 9):

$$\{a\ B{:}SW{:}W{:}N{:}NE\ b,\ a\ B{:}SW{:}W{:}E{:}SE\ c,\ a\ B{:}SW{:}W{:}E{:}SE\ d,$$
$$b\ B{:}S{:}SW{:}W{:}E{:}SE\ c,\ b\ S{:}SW\ d,\ d\ B{:}W{:}NW{:}N{:}NE{:}E\ c\}.$$

It is easy to verify that the spatial arrangement expressed by the above constraints is not realisable in $\mathbb{R}^2$, thus set C is inconsistent. Using only consistency-based composition we cannot discover this inconsistency: applying repeatedly consistency-based composition to the above set C will not derive any more cardinal direction constraints (and no contradiction!).

An algorithm that decides the consistency of a given set of basic cardinal direction constraints is presented in [40].

Unfortunately, we cannot use the cardinal direction relations defined in this paper in the constraint databases frameworks of [23] or [24]. In these frameworks, the class of constraints involved must be closed under the operation of variable elimination.

Definition 10. The operation of *variable elimination* takes as input a set C of constraints with set of variables X and a subset Y of X, and returns a new set of constraints C' such that $Sol(C') = \Pi_{X\setminus Y}(Sol(C))$ where Π_Z is the standard operation of projection of a relation on a subset Z of its set of columns.

Example 8 above also demonstrates that the class of cardinal direction constraints examined in this paper is not closed under variable elimination. For example, if we have constraints

$$a\ N\ b, \quad b\ N\ c$$

and we eliminate variable b, the result of the elimination is not expressible in the constraint language we started with! Similarly to the case of existential composition (Section 6), the language of Section 3 needs to be modified in order to define the operator of variable elimination and to be used in a constraint database model. Notice that to express variable elimination we may need to introduce additional relations than theses needed for existential composition. We are currently working towards extending the language of cardinal direction constraints to remove the above mentioned limitations.

8. Extensions

In Section 3 we have presented a model defining cardinal direction relations for the connected regions in *REG*. In this section we will present two interesting extensions of the basic model and discuss the composition problem for relations in these extensions.

The first extension is to modify the framework so that it also covers regions in REG^* [40]. As we have seen in Section 3, regions in REG^* can be disconnected and can have holes (Fig. 17).

The set of basic cardinal direction relations for regions in REG^* contains $\sum_{i=1}^{9}\binom{9}{i} = 511$ elements. We will use $\mathcal{D}^*$ to denote this set. Similarly to relation in $\mathcal{D}$, relations in $\mathcal{D}^*$ are jointly exhaustive and pairwise disjoint. Relations in $\mathcal{D}^*$ enables us to express spatial arrangements natural for geographical areas, e.g., $a\ S{:}W\ b$, $a\ S{:}N\ b$ and $a\ S{:}W{:}NW{:}N{:}E\ b$ (Fig. 17), that are not possible in the model of Section 3.

The results of Sections 4, 5 and 6 can be also applied for the composition of relations in $\mathcal{D}^*$ with the following changes.

- We use the function symbol δ^* in place of δ. Function symbol δ^* is defined similarly to δ. For arbitrary single-tile cardinal direction relations $R_1, \ldots, R_k$, the notation $\delta^*(R_1, \ldots, R_k)$ is a shortcut for the disjunction of all cardinal direction relations in $\mathcal{D}^*$

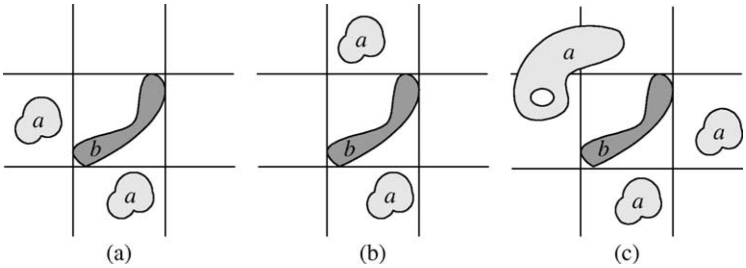

Fig. 17. Regions in REG^* and relations in $\mathcal{D}^*$.

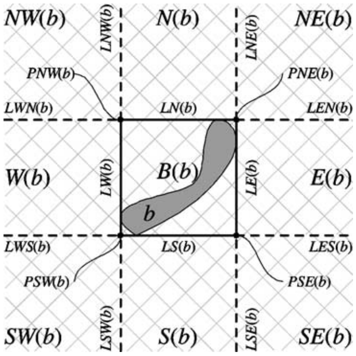

Fig. 18. Including points and lines.

that can be constructed by combining single-tile relations $R_1, \ldots, R_k$. For instance, $\delta^*(SW, W, NW)$ stands for the disjunctive relation:

$$\{SW,\ W,\ NW,\ SW{:}W,\ W{:}NW,\ SW{:}NW,\ SW{:}W{:}NW\}.$$

Notice that it is always $\delta(R_1, \ldots, R_k) \subseteq \delta^*(R_1, \ldots, R_k)$. For instance, $SW{:}NW \notin \delta(SW, W, NW)$ while $SW{:}NW \in \delta^*(SW, W, NW)$.
Similarly to δ, we define:

$$\delta^*\big(\delta^*(R_{11}, \ldots, R_{1k_1}), \delta^*(R_{21}, \ldots, R_{2k_2}), \ldots, \delta^*(R_{m1}, \ldots, R_{mk_m})\big)$$
$$= \delta^*(R_{11}, \ldots, R_{1k_1}, R_{21}, \ldots, R_{2k_2}, \ldots, R_{m1}, \ldots, R_{mk_m}).$$

- We replace every occurrence of $\mathcal{D}$ with $\mathcal{D}^*$. These replacements take place only in Theorem 2 and Algorithm COMPOSE.

The second extension is to accommodate any region in $\mathbb{R}^2$ (i.e., to include points and lines). Points and lines have been excluded carefully from *REG* (they are not homeomorphic to the unit disk), but they can be easily included by dividing the space around the reference region b into 25 areas as follows (see also Fig. 18).

- 9 two-dimensional areas ($B(b)$, $S(b)$, $SW(b)$, $W(b)$, $NW(b)$, $N(b)$, $NE(b)$, $E(b)$, $SE(b)$). They are formed form the axis of the bounding box of the reference region b (grey shaded areas of Fig. 18). Notice that each area does not include the parts of the axis forming it (contrary to the to the model of Section 3). These areas correspond to the bounding box and the 8 cardinal directions.
- 8 semi-lines ($LSW(b)$, $LWS(b)$, $LWN(b)$, $LNW(b)$, $LNE(b)$, $LEN(b)$, $LES(b)$, $LSE(b)$). These semi-lines are formed from the vertical and horizontal lines that start from the corners of the bounding box of the reference region b (dotted lines of Fig. 18). Notice that each semi-line does not include the corner of the bounding box.
- 4 line segments ($LS(b)$, $LW(b)$, $LN(b)$, $LE(b)$). These lines segments correspond to the sides of the bounding box of the reference region b (solid lines of Fig. 18). Notice that each line segment does not include the corners of the bounding box.

- 4 points $(PSW(b), PNW(b), PNE(b), PSE(b))$. These points correspond to the corners of the bounding box of the reference region b.

The above partition of the reference space should be contrasted to the partition of Section 3 that divides the space into 9 areas. The new set, denoted by $\mathcal{D}^{\mathbb{R}^2}$, contains $\sum_{i=1}^{25} \binom{25}{i} = 33{,}554{,}431$ jointly exhaustive and pairwise disjoint cardinal direction relations. The results of Sections 4, and 5 concerning consistency-based composition can be easily modified to handle this case as well. This extension is as in [20] (but the composition problem is ignored there!).

Note that the vocabulary of $\mathcal{D}^{\mathbb{R}^2}$ is much richer that the vocabulary of $\mathcal{D}$. As a result existential composition can be defined for a larger set of relations. For instance, in the vocabulary of $\mathcal{D}^{\mathbb{R}^2}$, it is $N; N = N$. This should be contrasted with the case of $\mathcal{D}$ where the result of $N; N$ cannot be defined in the available language (see Example 8). It is easy to verify that existential composition can be defined for all single-tile relations of $\mathcal{D}^{\mathbb{R}^2}$ (a 25×25 composition table can be constructed for this with some patience). Unfortunately, the result of existential composition cannot be defined for the whole set of $\mathcal{D}^{\mathbb{R}^2}$ unless the language is augmented with appropriate predicates. This is demonstrated by the following example.

Example 10. Consider region variables a, b, c and cardinal direction constraints

$$a\ S{:}LSW{:}SW{:}LSW{:}W\ b \quad \text{and} \quad b\ SW\ c$$

(see Fig. 19(a)). The only cardinal direction constraint implied by these two constraints is $a\ SW\ c$. Let us check whether $(S{:}LSW{:}SW{:}LSW{:}W; SW) = SW$ holds. If this equality was correct then for each pair of regions a_0, c_0 such that $a_0\ SW\ c_0$, there exists a region b_0 such that $a_0\ S{:}LSW{:}SW{:}LSW{:}W\ b_0$ and $b_0\ SW\ c_0$. However, Fig. 19(b) shows two such regions a_0 and c_0 such that $a_0\ SW\ c_0$ and it is impossible to find a region b_0 such that $a_0\ S{:}LSW{:}SW{:}LSW{:}W b_0$.

Similarly to Example 8, we notice that the given constraint on a and b implies the following constraint on a: the *area* covered by each region substituted for a cannot be rectangular; it should extend so that it covers tiles $S(b)$, $LSW(b)$, $SW(b)$, $LWS(b)$ and $W(b)$ for any region b. Intuitively, this constraint is not expressible in the language of cardinal direction relations $\mathcal{D}^{\mathbb{R}^2}$.

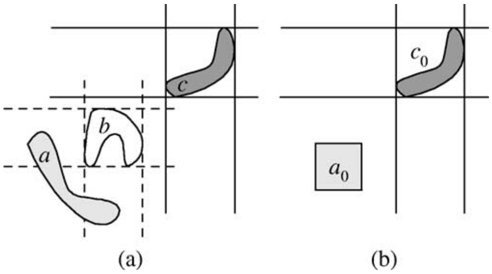

Fig. 19. Existential composition is not expressible even in $\mathcal{D}^{\mathbb{R}^2}$.

9. Conclusions

In this paper we gave a formal presentation of the cardinal direction model of Goyal and Egenhofer [19]. We used our formal framework to study the composition operation for cardinal direction relations in this model. We focused on two definitions of the composition operator, namely consistency-based and existential composition. We first showed that the method proposed in [19] to compute consistency-based composition does not always work correctly. Then, we considered progressively more expressive classes of cardinal direction relations and gave consistency-based composition algorithms for these classes. Our theoretical framework allowed us to prove formally that our algorithms are correct. Then, we considered existential composition and we have demonstrated that in some cases, the binary relation resulting from the composition of two cardinal direction relations cannot even be expressed using the vocabulary defined in [19]. Finally, we have presented some extensions to the basic model and sketched composition algorithms for these extensions.

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